Applied Numerical Mathematics 10 (1992) 139–147
North-Holland

APNUM304

Numerical experiments with a nonlinear evolution equation which exhibits blow-up

Kris Stewart and Tunc Geveci
Department of Mathematical Sciences, San Diego State University, San Diego, CA 92182, USA

Abstract

Stewart, K. and T. Geveci, Numerical experiments with a nonlinear evolution equation which exhibits blow-up, Applied Numerical Mathematics 10 (1992) 139–147.

The results of numerical experiments which involve a nonlinear evolution equation exhibiting blow-up and the use of spectral and pseudospectral methods are examined. It is observed that in spite of the expected exponential-order accuracy of these methods in the case of smooth solutions, they do not perform well in the detection of blow-up.

1. Background

Constanstin, Lax and Majda [4] have suggested a simple one-dimensional model for the three-dimensional vorticity equation:

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega, \qquad t > 0, \quad -\infty < x < \infty, \tag{1}$$

$$w(x, 0) = w_0(x),$$

where $\omega = \omega(x, t)$ and $H(\omega)$ is the Hilbert transform,

$$H(w)(x) = \frac{1}{\pi} \mathrm{PV} \int \frac{w(y)}{x-y} \, \mathrm{d}y$$

(PV denotes principal value).

We shall consider the periodic counterpart of (1), where $\omega(\cdot, t)$ is periodic with period 2π. In terms of the Fourier expansion of ω,

$$\omega(x) = \sum_{k=-\infty}^{\infty} \hat{\omega}_k \, \mathrm{e}^{ikx},$$

where

$$\hat{\omega}_k = \frac{1}{2\pi} \int_0^{2\pi} \omega(x) \, \mathrm{e}^{-ikx} \, \mathrm{d}x,$$

$$H(\omega)(x) = \sum_{k=-\infty}^{\infty} -\mathrm{i} \, \mathrm{sign}(k) \hat{\omega}_k \, \mathrm{e}^{ikx} \tag{2}$$

Correspondence to: K. Stewart, Department of Mathematical Sciences, San Diego State University, San Diego, CA 92182, USA.

where $\mathrm{i} = \sqrt{-1}$ and

$$\operatorname{sign}(k) = \begin{cases} 1, & \text{if } k = 1, 2, \dots, \\ 0, & \text{if } k = 0, \\ -1, & \text{if } k = -1, -2, \dots. \end{cases}$$

Let $H^1_{2\pi}$ denote the Sobolev space of square-integrable 2π-periodic functions with square-integrable first derivatives, equipped with the norm

$$\| \omega \|_1 = \left(\| \omega \|^2 + \| D_x \omega \|^2 \right)^{1/2}$$

where $\| \cdot \|$ denotes the L^2-norm,

$$\| \omega \|^2 = \int_0^{2\pi} | \omega(x) |^2 \, \mathrm{d}x.$$

In terms of the Fourier expansion of ω,

$$\| \omega \|^2 = 2\pi \sum_{k=-\infty}^{\infty} | \hat{\omega}_k |^2 \tag{3}$$

and

$$\| \omega \|_1^2 = 2\pi \sum_{k=-\infty}^{\infty} (1 + k^2) | \hat{\omega}_k |^2. \tag{4}$$

As noted in [4], (1) is an evolution equation in $H^1_{2\pi}$ with locally Lipschitz right-hand side, so that standard local existence and uniqueness results are applicable. In [4], the following explicit solution has been constructed:

$$\omega(x, t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2 \omega_0^2(x)}. \tag{5}$$

From (5) it is observed in [4] that the solution blows up in finite time if

$$Z = \{ x \mid \omega(x) = 0 \text{ and } H\omega_0(x) > 0 \} \tag{6}$$

is not empty. In this case, $\omega(x, t)$ becomes infinite as $t \nearrow T_*$, where the blow-up time

$$T_* = 2/M, \quad M = \sup\{ (H\omega_0)_+(x) \colon \omega_0(x) = 0 \}.$$

In particular, if $\omega_0(x) = \cos \pi x$, $(H\omega_0)(x) = \sin \pi x$ and

$$\omega(x, t) = \frac{4 \cos \pi x}{(2 - t \sin^2 \pi x) + t^2 \cos^2 \pi x} \tag{7}$$

for which the blow-up time $T_* = 2$. $\omega(x, t)$ develops a singularity like $1/(x - \frac{1}{2})$ near $x = \frac{1}{2}$ as $t \nearrow 2$ as shown in Fig. 1. Although the problem is posed for $-\infty < x < \infty$, Fig. 1 only presents the solution on a subset of the spatial domain symmetric about the singularity point.

As observed in [4] with $\omega(x, t)$ given by (7),

$$\| \omega(\cdot, t) \| \nearrow \infty \quad \text{as } t \nearrow 2$$

and of course

$$\| \omega(\cdot, t) \|_1 \nearrow \infty \quad \text{as } t \nearrow 2.$$

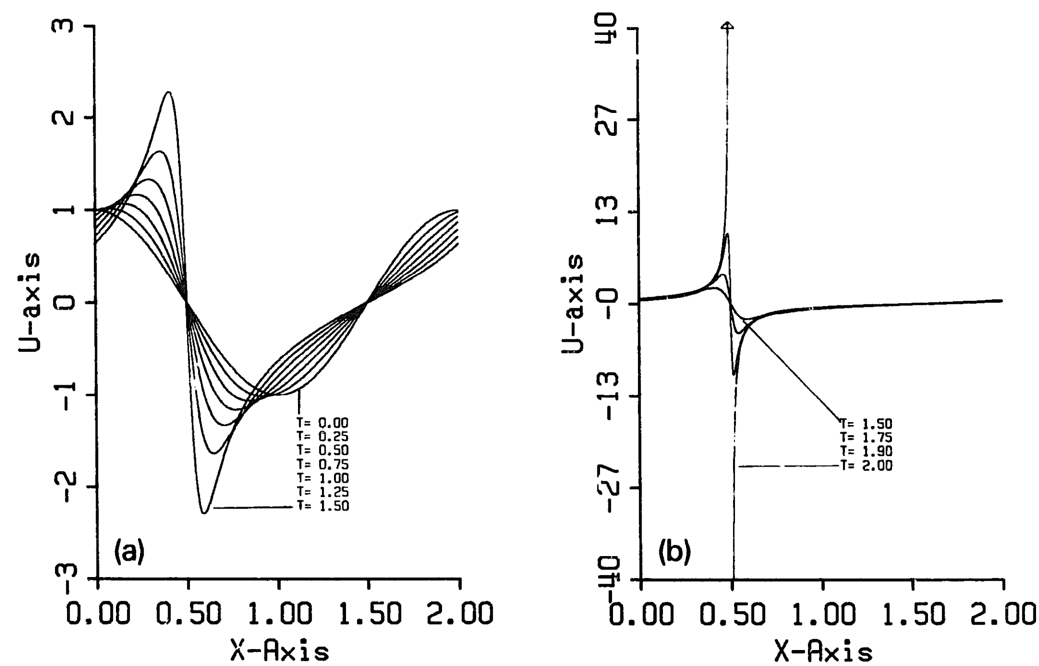

Fig. 1. (a) The evolution of the exact solution for $t = 0$ up to 1.5. (b) The evolution of the exact solution for t from 1.5 to 2.0.

The existence of an explicit solution which exhibits blow-up and the convenient description of the Hilbert transform in terms of Fourier series (2) make this equation an ideal model to demonstrate what may happen if either spectral or pseudospectral spatial discretization schemes are used to obtain a numerical approximation to a nonlinear evolution equation. This is the aim of this paper.

2. The approximation schemes and numerical results

As a general reference on spectral and pseudospectral methods, the books by Gottlieb and Orszag [7] and Canuto, Hussaini, Quarteroni and Zang [3] may be consulted.

The approximate solutions will be sought in the form of trigonometric polynomials

$$w(x) = \sum_{k=-N}^{N-1} \tilde{w}_k \, e^{ikx}. \tag{8}$$

If we set

$$x_j = j\pi/N, \quad j = 0, 1, \ldots, 2N-1, \qquad w_j = w(x_j),$$

we have

$$\tilde{w}_k = \frac{1}{2N} \sum_{j=0}^{2N-1} w_j \, e^{-ikx_j}, \quad k = -N, \ldots, N-1.$$

Denoting

$$\tilde{W} = \left[\tilde{w}_{-N}, \ldots, \tilde{w}_{N-1}\right]^{\mathrm{T}}, \qquad W = \left[w_0, \ldots, w_{2N-1}\right]^{\mathrm{T}},$$

W and $\tilde{W}$ are thus related by the discrete Fourier transform and its inverse:

$$\mathscr{F}(W) = \tilde{W}, \qquad \mathscr{F}^{-1}(\tilde{W}) = W. \tag{9}$$

The spectral approximation for the solution to the nonlinear equation (1),

$$\omega^N(x, t) = \sum_{k=-N}^{N-1} \tilde{\omega}_k^N(t)\, \mathrm{e}^{ikx}, \tag{10}$$

is based on the Galerkin idea [3, p. 77] of substituting the trigonometric approximation (10) into the differential equation (1) and projecting onto the basis functions $q_l(x) = \mathrm{e}^{ilx}$, yielding

$$\left\langle \frac{\partial \omega^N}{\partial t}(\cdot, t) - H\omega^N(\cdot, t), q_l \right\rangle = 0, \quad l = -N, \ldots, N-1,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2-inner product.

This results in a system of ordinary differential equations (ODEs) which can be expressed as follows: Let

$$\tilde{\Omega}^N = \left[\tilde{\omega}_{-N}^N, \ldots, \tilde{\omega}_{N-1}^N\right]^{\mathrm{T}}.$$

Define $G^N: \mathbb{R}^{2N} \to \mathbb{R}^{2N}$, where the kth component is expressed as the convolution

$$\left(G^N(\tilde{\Omega}^N)\right)_k = \sum_{\substack{p+q=k \\ -N \leqslant p,q \leqslant N-1}} -\mathrm{i}\,\mathrm{sign}(p)\, \tilde{\omega}_p^N \tilde{\omega}_q^N.$$

The system of ODEs is

$$\frac{\mathrm{d}\tilde{\Omega}^N}{\mathrm{d}t}(t) = G^N\left(\tilde{\Omega}^N(t)\right), \tag{11}$$

with initial condition

$$\tilde{\Omega}^N(0) = \left[\tilde{\omega}_{0,-N}^N, \ldots, \tilde{\omega}_{0,N-1}^N\right]^{\mathrm{T}}$$

where

$$\tilde{\omega}_{0,k}^N = \frac{1}{2N} \sum_{j=0}^{2N-1} \omega_0(x_j)\, \mathrm{e}^{-ikx_j}, \quad k = -N, \ldots, N-1.$$

The pseudospectral approximation is again in the form (10) but is determined by the collocation condition [3, p. 79] of satisfying the differential equation (1) at the points used to define the trigonometric representation, $x_j = j\pi/N$:

$$\frac{\mathrm{d}\omega^N}{\mathrm{d}t}(x_j, t) = H\omega^N(x_j, t) \cdot \omega(x_j, t), \quad j = 0, \ldots, 2N-1,$$

with initial conditions

$$\omega^N(x_j, 0) = \omega_0(x_j), \quad j = 0, \ldots, 2N-1.$$

This leads to the ODE system

$$\frac{\mathrm{d}\Omega^N}{\mathrm{d}t} = \mathscr{G}(\Omega^N(t)), \tag{12}$$

with initial condition

$$\Omega^N(0) = [\omega_0(0), \omega_0(x_1), \ldots, \omega_0(x_{2N-1}]^{\mathrm{T}},$$

where the jth component of $\mathscr{G}(\Omega^N)$ is obtained as the product of ω_j^N and the jth component of $(\mathscr{F}^{-1}\mathscr{H}\mathscr{F})(\Omega^N)$, $\mathscr{H}$ being the operation corresponding to the Hilbert transform (2), namely multiplication of the kth Fourier coefficient by i sign(k).

3. Numerical results

Numerical solutions of the ODEs from the spectral (11) and pseudospectral (12) spatial discretizations were computed with initial condition $\omega_0(x) = \cos \pi x$ for $x \in [0, 2]$. The solutions were computed using the NCAR FFT package [9] to compute the discrete Fourier transfoims (9) and a fifth-order, variable-stepsize Runge–Kutta–Fehlberg solver, RKF45 [8] (also in [6]) to advance the solution in time. The tolerance for the ODE solver was selected in order to display the accuracy of the spatial approximation faithfully, an absolute error tolerance of 10^{-6} proved adequate. Prior to blow-up, the accuracy of the spectral and pseudospectral

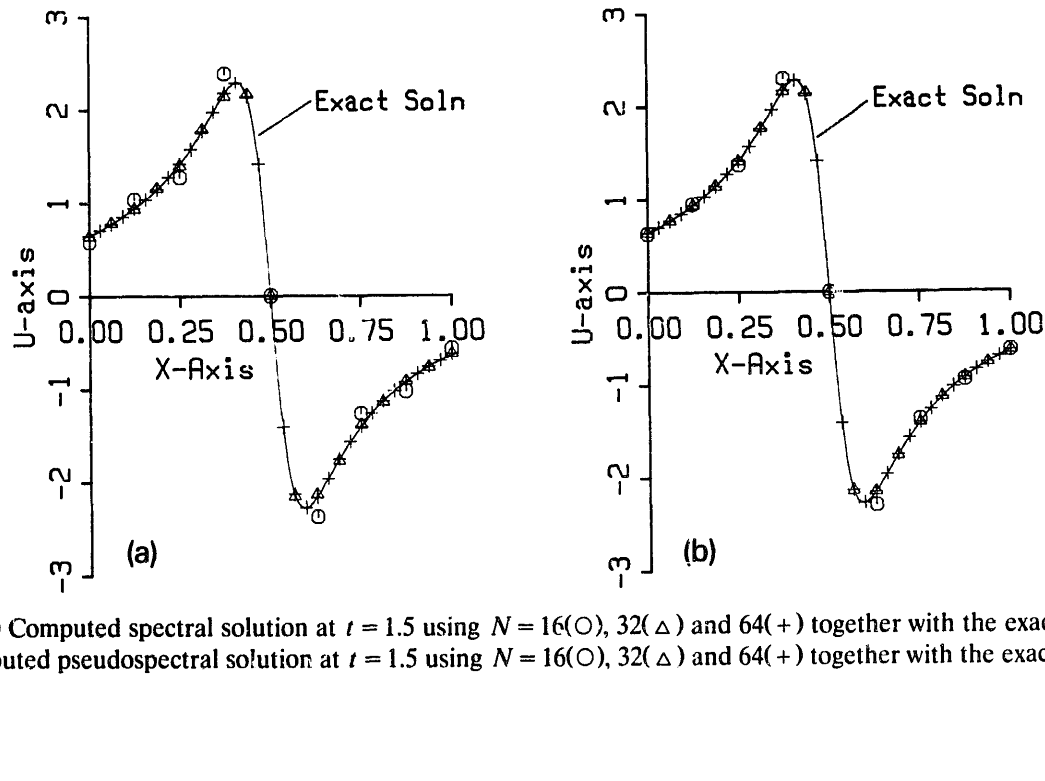

Fig. 2. (a) Computed spectral solution at $t = 1.5$ using $N = 16(\bigcirc)$, $32(\triangle)$ and $64(+)$ together with the exact solution. (b) Computed pseudospectral solution at $t = 1.5$ using $N = 16(\bigcirc)$, $32(\triangle)$ and $64(+)$ together with the exact solution.

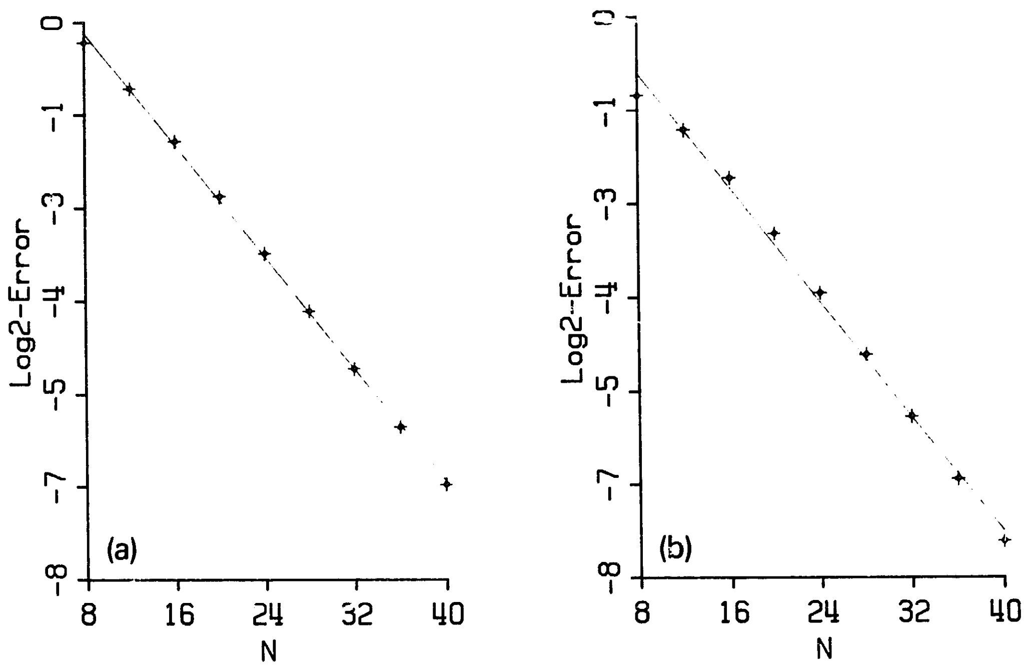

Fig. 3. (a) Log(error of spectral approximation) versus N at $t = 1.5$ for $N = 8(4)40$ together with linear regression fit. (b) Log(error of pseudospectral approximation) versus N at $t = 1.5$ for $N = 8(4)40$ together with linear regression fit.

schemes is as anticipated. Figure 2 exhibits the spectral and pseudospectral solutions at time $t = 1.5$, clearly demonstrating the increased accuracy as N increases from 16 up to 64 points. Although the problem was solved on the spatial domain [0, 2], only the portion [0,1], symmetric about the singularity point, is presented in the figures.

Tadmor [10] discussed the accuracy of spectral and pseudospectral approximation schemes in approximating analytic functions. He derived results which established exponential-order accuracy. The solution we are considering is meromorphic and we have verified exponential accuracy of both schemes prior to blow-up numerically. For example at $t = 1.5$, when we plot $\log_2(\| \omega_{\text{exact}}(\cdot, 1.5) - \omega_{\text{approx}}(\cdot, 1.5) \|)$ versus N we obtain the linear relationship as shown in Fig. 3. For clarity a straight line has also been fit to the data points using least squares.

On the other hand, neither scheme performs well in predicting the blow-up at time $t = 2$. The performance of the pseudospectral method is especially disappointing. As seen in Fig. 4, the method computes an "approximate solution" beyond $t = 2$ without a clear indication of blow-up. The calculations with the spectral method are more indicative of blow-up at $t = 2$, as seen in Fig. 5. The oscillations appear before $t = 2$ and become very pronounced immediately past $t = 2$, even though an accurate estimation of the actual blow-up time is not feasible. One observes that the use of the larger value of N results in some indication of the blow-up in that there are much wilder fluctuations.

As a diagnostic tool, one can monitor the growth of the L_2- or $H^1_{2\pi}$-norm of the discrete solution, given by (3) and (4) with the infinite sums replaced by the finite sum as in (8). Figures 6 and 7 display these norms for the pseudospectral and spectral methods, respectively. The scale for the pseudospectral results in Fig. 6 is significantly smaller than that for the spectral results. It is again clear that the pseudospectral method does not perform well, and that the

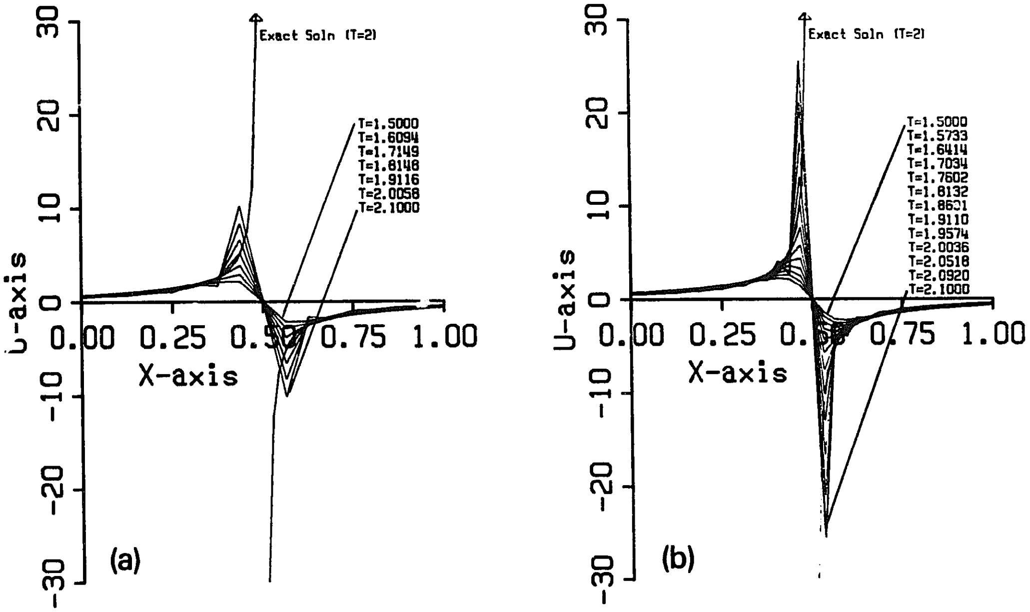

Fig. 4. (a) Computed pseudospectral solution for t from 1.5 up to 2.1 using $N = 32$ together with the exact solution. (b) Computed pseudospectral solution for t from 1.5 up to 2.1 using $N = 64$ together with the exact solution.

spectral method gives a clearer indication of blow-up. It is also clear that the growth of the $H^1_{2\pi}$-norm gives a much clearer indication of blow-up than the L_2-norm. This is not surprising in view of the fact that the evolution equation (1) is an equation with locally Lipschitz right-hand side in terms of the $H^1_{2\pi}$-norm.

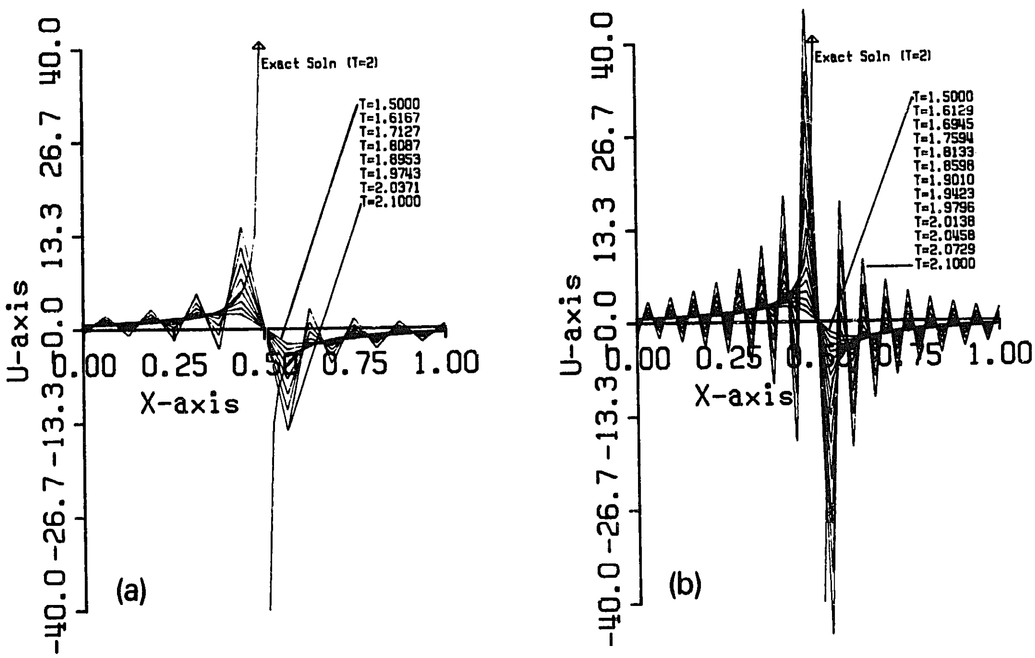

Fig. 5. (a) Computed spectral solution for t from 1.5 up to 2.1 with $N = 32$ together with the exact solution. (b) Computed spectral solution for t from 1.5 up to 2.1 with $N = 64$ together with the exact solution.

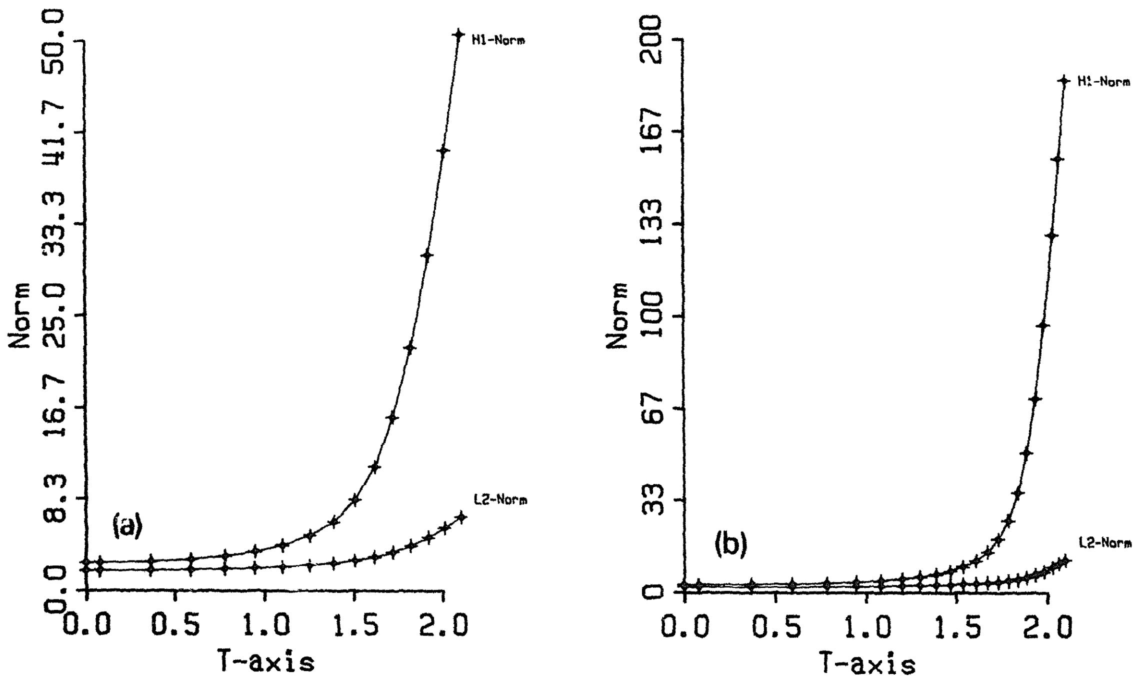

Fig. 6. (a) L_2- and $H^1_{2\pi}$-norm of the pseudospectral solution for t up to 2.1 using $N = 32$. (b) L_2- and $H^1_{2\pi}$-norm of the pseudospectral solution for t up to 2.1 using $N = 64$.

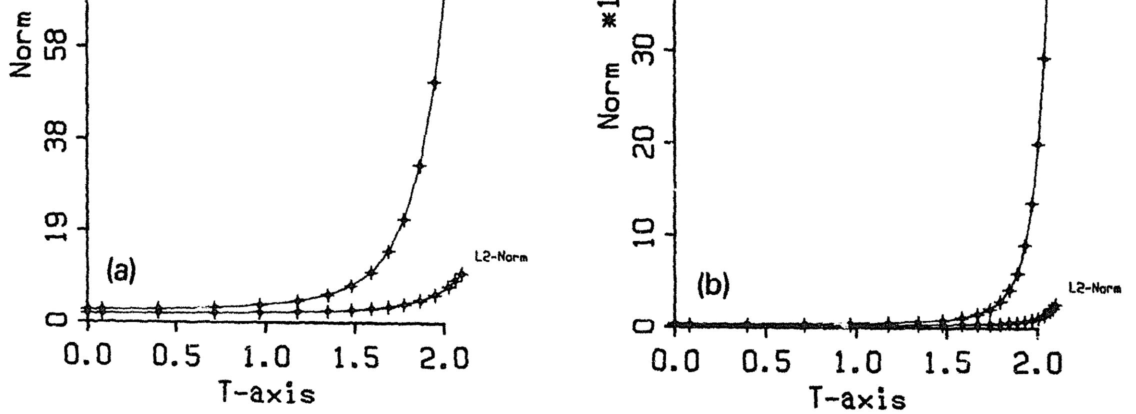

Fig. 7. (a) L_2- and H^1_{π}-norm of the spectral solution for t up to 2.1 using $N = 32$. (b) L_2- and $H^1_{2\pi}$-norm of the spectral solution for t up to 2.1 using $N = 64$.

4. Conclusion

In this note we have tried to demonstrate the difficulty of assessing the results obtained by using spectral and pseudospectral spatial approximation schemes coupled with a widely used ODE solver, in dealing with solutions which exhibit blow-up. The accuracy of these schemes is well established in the case of smooth solutions. We have verified this prior to blow-up in the model equation we have considered.

The spectral method does a much better job in indicating blow-up compared to the pseudospectral method. Aside from the oscillations in the solution obtained by the spectral method, the growth of the appropriate norm, in this case the $H^1_{2\pi}$-norm, is a convenient diagnostic tool.

The reader may consult [1,2,5] which include demonstrations of the effect of the instability of solutions to certain evolution equations of the Korteweg–de Vries and Benjamin–Ono type on the performance of certain approximation schemes.

It is thus obvious that one has to be very cautious in using standard schemes when such solutions are involved. Special schemes which are appropriate for the particular problem at hand need to be devised.

References

[1] J. Bona, V. Douglas and O. Karakashian, Fully discrete Galerkin methods for the Korteweg–de Vries equation, *Comput. Math. Appl.* 12A (1986) 859–884.

[2] J. Bona, P. Souganidis and W. Strauss, Stability and instability of solitary waves of Korteweg–de Vries type, *Proc. Roy Soc. Lond.* A411 (1987) 395–412.

[3] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods in Fluid Dynamics* (Springer, Berlin 1988).

[4] P. Constantin, P. Lax and A. Majda, A simple one-dimensional model for the three-dimensional vorticity equation, *Comm. Pure Appl. Math.* 38 (1985) 715–724.

[5] B. Fornberg and G. Whitham, A numerical and theoretical study of certain nonlinear wave phenomena, *Phil. Trans. Roy. Soc. Lond.* A289 (1978) 373–404.

[6] G. Forsythe, M. Malcolm and C. Moler, *Computer Methods for Mathematical Computations* (Prentice-Hall, Englewood Cliffs, NJ, 1977).

[7] D. Gottlieb and S. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications* (SIAM-CBMS, Philadelphia, PA, 1977).

[8] L. Shampine and H. Watts, Practical solution of ordinary differential equations by Runge–Kutta methods, Tech. Rept. SAND76-0585, Sandia National Laboratories (1975).

[9] P. Swarztrauber, FFTPACK, National Center for Atmospheric Research (4 ed., 1985), available from Argonne National Laboratories through Nanet.

[10] E. Tadmor, The exponential accuracy of Fourier and Chebyshev differencing methods, *SIAM J. Numer. Anal.* 23 (1986) 1–10.